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Further Investigation on Feasibility of Mathematical Programs with Equilibrium Constraints

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Abstract—In this paper, we present feasibility conditions for mathematical programs with affine equilibrium constraints (MPECs) where additional joint constraints are present that must be satisfied by the state and design variables of the problems. We show that these conditions are also sufficient for quadratic programming subproblems arising from the penalty interior point algorithm (PIPA) and the smooth sequential quadratic programming (SQP) algorithm for solving MPECs to be consistent.

Keywords—Mathematical programs with equilibrium constraints, Feasibility conditions.

1. INTRODUCTION

Let $f : R^{n+m} \rightarrow R$ be a continuously differentiable function, $A \in R^{p\times n}$, $B \in R^{p\times m}$, $N \in R^{m\times n}$, and $M \in R^{m\times m}$ be given matrices, $b \in R^p$ and $q \in R^m$ be given vectors, and C be a polyhedral cone in R^n. Consider the following mathematical program with affine equilibrium constraints where additional joint linear constraints are presents besides the equilibrium constraints:

$$\begin{aligned} \text{minimize} \quad & f(x,y) \\ \text{subject to} \quad & Ax + By = b, \\ & w = Nx + My - q, \\ & w^\top y = 0, \qquad x \in C,\ y \geq 0,\ w \geq 0. \end{aligned} \tag{1.1}$$

In (1.1), the variable x is called the design variable or the first-level variable and the variables y and w are called the state variables or the second-level variables. The equilibrium constraints (also called state constraints) are expressed in the form of a parametric linear complementarity problem (LCP) with x as the parameter and y as the primary variable. The additional joint constraint between the design variable x and the state variable y is assumed to be modeled by the linear equation $Ax + By = b$.

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The purpose of this paper is to further study the feasibility of problem (1.1) on the basis of the work by Fukushima and Pang [1], where the matrix B admits being nonzero. In the case where $B = 0$, the feasibility of (1.1) is relatively easy, we refer to Section 1 of [1] for some explanations. In the case where $B \neq 0$, the feasibility of MPEC (1.1) has been the major bottleneck for a complete treatment of problem (1.1) [1]. In this paper, we present sufficient conditions for problem (1.1) to be feasible. We show that these conditions are also sufficient to guarantee the consistence of the subproblems arising from the penalty interior point algorithm (PIPA) [2] and the smooth sequential quadratic programming (SQP) algorithm [3] for solving MPECs.

2. FEASIBILITY CONDITIONS FOR MPECS

In this section, we introduce sufficient conditions to ensure the feasibility of problem (1.1). We first rewrite the constraint system of (1.1) as follows.

$$\begin{aligned} Ax + By &= b, \\ y &\geq 0, \\ Nx + My - q &\geq 0, \\ x &\in C, \\ y^\top (Nx + My - q) &= 0. \end{aligned} \tag{2.1}$$

Then the MPEC (1.1) is feasible if and only if (2.2) is consistent. To study the consistence of system (2.2), we propose the following two assumptions.

(A_1) The linear system

$$\begin{aligned} Nx + My - q &\geq 0, \\ y &\geq 0, \\ x &\in C, \end{aligned} \tag{2.2}$$

is consistent.

(A_2) For any (u, v, s) satisfying

$$\begin{aligned} A^\top u + N^\top v &\in C^*, \\ v \circ \left(B^\top u + M^\top v\right) &\leq 0, \\ s \circ \left(B^\top u + M^\top v\right) &\geq 0, \\ (v + s) \circ s \geq 0, &\qquad s \geq 0, \end{aligned} \tag{2.3}$$

we have $(v + s)^\top (B^\top u + M^\top v) \geq (1/2)\, u^\top u$.

Conditions (A_1) and (A_2) are similar to the conditions given by Fukushima and Pang [1]. But both do not imply each other. As follows, we first provide some sufficient conditions for Assumption (A_2) to hold.

PROPOSITION 2.1. *Let*

$$P_1 \equiv \begin{pmatrix} -I & B \\ B^\top & M + M^\top \end{pmatrix}, \qquad P_2 \equiv -2 \begin{pmatrix} B^\top & 0 \\ 0 & M^\top \end{pmatrix},$$

and $\mathcal{K}$ denote the cone $\{(u, v) \in R^{p+m} : A^\top u + N^\top v \in C^*\}$. If for any $(u, v, s) \in \mathcal{K} \times R^m_+$, P_1 and P_2 satisfy the following inequality:

$$\left(u^\top v^\top\right) P_1 \begin{pmatrix} u \\ v \end{pmatrix} \geq \left(s^\top s^\top\right) P_2 \begin{pmatrix} u \\ v \end{pmatrix},$$

where R^m_+ denotes the nonnegative orthant of R^m. Then Assumption (A_2) holds.

It is easy to prove the above proposition, we omit it.

Next, we state the main theorem in this paper.

THEOREM 2.1. *Under Assumptions* (A_1) *and* (A_2), *system (2.1) is consistent.*

PROOF. We only require to prove that the following quadratic program respect to the variable $(x, y) \in R^{n+m}$ has an optimal solution, and that the minimum value of the objective function is zero.

$$\begin{array}{ll} \text{minimize} & y^\top (Nx + My - q) + \frac{1}{2}(Ax + By - b)^\top (Ax + By - b) \\ \text{subject to} & (2.2). \end{array} \tag{2.4}$$

Let $(\bar{x}, \bar{y})$ be a solution of (2.4). Then, there exist some multiplier vector $\mu \in R^m$ such that

$$\begin{aligned} &0 \le \bar{y} \perp \left(N\bar{x} + M\bar{y} + M^\top \bar{y} - q + B^\top B\bar{y} + B^\top A\bar{x} - B^\top b - M^\top \mu\right) \ge 0, \\ &C \ni \bar{x} \perp \left(N^\top \bar{y} + A^\top A\bar{x} + A^\top B\bar{y} - A^\top b - N^\top \mu\right) \in C^*, \\ &0 \le \mu \perp (N\bar{x} + M\bar{y} - q) \ge 0. \end{aligned} \tag{2.5}$$

Let $\phi = N\bar{x} + M\bar{y} - q$ and $\psi = A\bar{x} + B\bar{y} - b$. Then (2.5) can be written as:

$$\begin{aligned} &0 \le \bar{y} \perp \left(\phi + B^\top \psi + M^\top (\bar{y} - \mu)\right) \ge 0, \\ &C \ni \bar{x} \perp \left(A^\top \psi + N^\top (\bar{y} - \mu)\right) \in C^*, \\ &0 \le \mu \perp \phi \ge 0. \end{aligned}$$

It then follows that

$$\begin{aligned} \mu \circ \phi &= 0, \\ \bar{y} \circ \left(\phi + B^\top \psi + M^\top (\bar{y} - \mu)\right) &= 0, \\ \mu \circ \left(\phi + B^\top \psi + M^\top (\bar{y} - \mu)\right) &\ge 0, \\ \bar{y} \circ \phi &\ge 0, \\ \bar{y} \circ \mu &\ge 0. \end{aligned} \tag{2.6}$$

So, we can deduce

$$\begin{aligned} (\bar{y} - \mu) \circ \left(B^\top \psi + M^\top (\bar{y} - \mu)\right) &\le (\bar{y} - \mu) \circ \left(\phi + B^\top \psi + M^\top (\bar{y} - \mu)\right) \\ &\le -\mu \circ \left(\phi + B^\top \psi + M^\top (\bar{y} - \mu)\right) \\ &\le 0 \end{aligned}$$

and

$$\mu \circ \left(B^\top \psi + M^\top (\bar{y} - \mu)\right) = \mu \circ \left(\phi + B^\top \psi + M^\top (\bar{y} - \mu)\right) \ge 0.$$

Let $u = \psi$, $v = \bar{y} - \mu$, and $s = \mu$. Then (u, v, s) satisfies (2.3). So, it follows from (A_2) that

$$(v + s)^\top \left(B^\top u + M^\top v\right) \ge \frac{1}{2} u^\top u. \tag{2.7}$$

From the second equation in (2.3), we have

$$\begin{aligned} (\bar{y})^\top \phi + \frac{1}{2}\psi^\top \psi &= -(\bar{y})^\top \left(B^\top \psi + M^\top (\bar{y} - \mu)\right) + \frac{1}{2}\psi^\top \psi \\ &= -(v + s)^\top \left(B^\top u + M^\top v\right) + \frac{1}{2} u^\top u \\ &\le 0, \end{aligned}$$

where the last inequality comes from (2.7). Also from the fourth inequality in (2.3), we have $(\bar{y})^\top \phi + (1/2)\psi^\top \psi \ge 0$, so we obtain that

$$(\bar{y})^\top \phi + \frac{1}{2}\psi^\top \psi = 0.$$

Since the objective function in (2.4) can be rewritten as $(\bar{y})^\top \phi + (1/2)\psi^\top \psi$, we have proved that it attains minimum value zero at $(\bar{x}, \bar{y})$, thus, the desired result holds. ∎

3. FEASIBILITY OF SUBPROBLEM OF PIPA OR SQP

In this section, we show that Conditions (A_1) and (A_2) are also sufficient to guarantee the consistence of the subproblems arising from PIPA [2] and the smooth SQP algorithm [3] for solving MPEC (1.1). The subproblem of the PIPA is the following quadratic program

$$\begin{aligned} \text{minimize} \quad & \nabla f\left(x^k, y^k\right)^\top \begin{pmatrix} dx \\ dy \end{pmatrix} + \frac{1}{2}\left(dx^\top, dy^\top, dw^\top\right) Q_k \begin{pmatrix} dx \\ dy \\ dw \end{pmatrix} \\ \text{subject to} \quad & A\,dx + B\,dy = 0, \\ & N\,dx + M\,dy - dw = 0, \\ & W^k\,dy + Y^k\,dw = -Y^k w^k + \sigma_k \mu_k e, \\ & x^k + dx \in C, \end{aligned} \tag{3.1}$$

where $Q_k \in R^{(n+2m)\times(n+2m)}$ is a symmetric positive definite matrix, $W^k = \operatorname{diag}(w_i^k)$, $Y^k = \operatorname{diag}(y_i^k)$, $\sigma_k \in (0,1)$ is the centerizing stepsize that control the direction (dx^k, dy^k, dw^k) between the pure Newton direction ($\sigma = 0$) and the perfectly central direction ($\sigma = 1$), and μ_k is the positive scalar given by $\mu_k \equiv (w^k)^\top y^k/m$. We refer to Chapter 6 of [2] for details about PIPA.

The following theorem shows that Conditions (A_1) and (A_2) are sufficient to ensure the consistence of (3.1).

THEOREM 3.1. *Under Assumptions (A_1) and (A_2), the quadratic program (3.1) is consistent for all k.*

PROOF. Let $t = vx^k + dx$. It then follows from (3.1) that

$$\begin{aligned} At + B\,dy &= Ax^k, \\ Nt + \left(M + \left(Y^k\right)^{-1} w^k\right) dy &= s^k, \\ t &\in C, \end{aligned} \tag{3.2}$$

where $s^k \equiv Nx^k - w^k + \sigma_k \mu_k (Y^k)^{-1} e$.

By a generalized Farkas lemma [4], it is clear that for an arbitrary s^k, system (3.2) is consistent if and only if the implication

$$\left.\begin{aligned} A^\top u + N^\top v \in C^* \\ B^\top u + \left(M^\top + \left(Y^k\right)^{-1} W^k\right)^\top v = 0 \end{aligned}\right\} \Rightarrow \left(Ax^k\right)^\top u + \left(s^k\right)^\top v \geq 0 \tag{3.3}$$

holds. The arbitrariness of s^k implies that the right-hand side of implication (3.3) is equivalent to

$$\left(Ax^k\right)^\top u \geq 0, \qquad v = 0. \tag{3.4}$$

Let (u, v) satisfy the left-hand side of implication (3.3). Then $B^\top u + M^\top v = -(Y^k)^{-1} W^k v$. Therefore, we have

$$v \circ \left(B^\top u + M^\top v\right) = -v \circ \left(\left(Y^k\right)^{-1} W^k v\right) \leq 0, \tag{3.5}$$

because Y^k abd W^k are diagonal matrices with positive entities. Under Assumptions (A_1) and (A_2), we get from (3.5) and Theorem 2.1 that $u = 0$ and $v^\top M^\top v \geq 0$ by letting $s = 0$ in (2.3). Consequently, (3.5) implies $v^\top (Y^k)^{-1} W^k v = 0$. By the positive definiteness of $(Y^k)^{-1} W^k$, we obtain $v = 0$. Thus, the implication (3.3) holds. This completes the proof. ∎

Now, consider the subproblem of the smooth SQP algorithm for MPEC (1.1) (see [3]):

$$\begin{array}{ll}\text{minimize} & \nabla f\left(x^k, y^k\right)^\top \begin{pmatrix} dx \\ dy \end{pmatrix} + \frac{1}{2}\left(dx^\top, dy^\top, dw^\top\right) Q_k \begin{pmatrix} dx \\ dy \\ dw \end{pmatrix} \\ \text{subject to} & A\,dx + B\,dy = 0, \\ & N\,dx + M\,dy - dw = 0, \\ & D_a^k\,dy + D_b^k\,dw = -\Phi\left(y^k, w^k, \mu_k\right), \\ & x^k + dx \in C, \end{array} \tag{3.6}$$

where $Q_k \in R^{(n+2m)\times(n+2m)}$ is a symmetric positive definite matrix,

$$D_a^k = \operatorname{diag}\left(\frac{\partial \phi\left(y_i^k, qw_i^k, \mu_k\right)}{\partial a}\right), \qquad D_b^k = \operatorname{diag}\left(\frac{\partial \phi\left(y_i^k, w_i^k, \mu_k\right)}{\partial b}\right),$$

$$\text{and} \qquad \Phi(y, w, \mu) \equiv \begin{pmatrix} \phi\left(y_1, w_1, \mu\right) \\ \phi\left(y_2, w_2, \mu\right) \\ \cdots \\ \phi\left(y_m, w_m, \mu\right) \end{pmatrix}$$

with $\phi(a, b, \mu) \equiv a + b - \sqrt{a^2 + b^2 + \mu}$.

In a way similar to Theorem 3.1, it is not difficult to prove the following theorem.

THEOREM 3.2. *Under Assumptions* (A_1) *and* (A_2)*, the quadratic program (3.6) is feasible for all* k.

4. CONCLUSIONS

The results presented in this paper show that Conditions (A_1) and (A_2) not only guarantee the feasibility of MPEC (1.1), but also can be used as the conditions to extend the convergence results for PIPA [2] and the smooth SQP method [3] such that these algorithms are applicable for the case that $B \neq 0$. However, it still deserves further investigation to find some more easily verified conditions for Assumption (A_2) to hold in practical applications (see Proposition 2.1).

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